Supplementary Notes for Classical Mechanics and Electromagnetism in Accelerator Physics

The US Particle Accelerator School
June 10-21, 2013
Fort Collins, Colorado

Gregory Penn

version June 19, 2013

Supplementary Notes 1

Extended Hamiltonian and Canonical Transformations

1.1 Generalizing the Independent Co-ordinate

The variational principle on which the Lagrangian is based is that

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\ \mathrm{d}t \tag{1.1}$$

takes an extremum when $q_i(t)$ matches its true trajectory, keeping the constraint that the end points are fixed. Here, time takes on a special role which is somewhat inconvenient when the system is explicitly time-dependent, or when considering relativistic dynamics.

To put time on a more equal footing with other co-ordinates, we consider an almost arbitary quantity τ which characterizes the motion of the system. The only requirement is that in some neighborhood around the actual evolution or path of the system, any slightly altered path has to have well-defined values of τ which vary monotonically along the path (no double values, at least for short sections of the path). Potentially useful choices for τ are the proper time, path length, one of the spatial co-ordinates if the geometry allows it, or a rescaling of the time (this can be an alternative to using the Floquet transformation). Note that τ is in some sense a local function of the q_i, derivatives $\dot{q}_i$, and time t, but we will focus on the inverse relationships $q_i(\tau)$, $\dot{q}_i(\tau)$, and $t(\tau)$ defined along a given path. We can then rewrite the action integral as

$$S = \int_{\tau_1}^{\tau_2} L\left[q_i(\tau), \frac{q_i'(\tau)}{t'(\tau)}, t(\tau)\right] t'(\tau)\ \mathrm{d}\tau\ , \tag{1.2}$$

where $t' \equiv \mathrm{d}t/\mathrm{d}\tau$ and $q_i' \equiv \mathrm{d}q_i/\mathrm{d}\tau = t'\mathrm{d}q_i/\mathrm{d}t = t'\dot{q}_i$. Now we can treat time t as just another co-ordinate, on an equal footing with the others. This gives us a new Lagrangian with an extra degree of freedom,

$$L_{\mathrm{ext}}(q_i, t, q_i', t') = L\left(q_i, \frac{q_i'}{t'}, t\right) t'\ . \tag{1.3}$$

There is never any explicit dependence on τ, which makes sense because it is arbitrary. Of course, there are extra constraints in this problem, but they fit into the Lagrangian framework. In particular, requiring $t(\tau_1) = t_1$ is no different than imposing a boundary condition on the value of $q_i(\tau_1)$. Now if we recalculate the canonical momentum, the term conjugate to q_i is still the same p_i as in the original formulation, but there is a new quantity

$$p_t \equiv \frac{\partial L_{\text{ext}}}{\partial t'} = \frac{1}{t'}\left(-\sum_i p_i q_i' + L_{\text{ext}}\right) = -\sum_i p_i \dot{q}_i + L(q_i, \dot{q}_i, t) = -H(q_i, p_i, t) \ . \tag{1.4}$$

Normally, there would not be a constraint that one value of momentum is determined by the co-ordinates and the other momenta. This is because we added the extra degree of freedom and have to take it away somehow. Thus, even though the action can be rewritten as

$$S = \int_{\tau_1}^{\tau_2} \left(\sum_i p_i q_i' + p_t t'\right) \, \mathrm{d}\tau \ , \tag{1.5}$$

which suggests that the extended Hamiltonian is equal to zero, we have to impose the constraint that

$$p_t + H(q_i, p_i, t) = 0 \tag{1.6}$$

in order to obtain the equations of motion. This is an example of an auxiliary condition. Auxiliary conditions are handled by adding the left hand side times some arbitary function of τ to the functional form of the original Hamiltonian, but since in this case we are starting from zero we just have

$$H_{\text{ext}} = t'(\tau)\left[p_t + H(q_i, p_i, t)\right] \ . \tag{1.7}$$

Then the equations of motion are unchanged for $\mathrm{d}p_i/\mathrm{d}t$, but the evolution of p_t is

$$p_t' = -\frac{\partial H_{\text{ext}}}{\partial t} = -t'\frac{\partial H}{\partial t} \ . \tag{1.8}$$

Given that $\mathrm{d}H/\mathrm{d}t = \partial H/\partial t$, this guarantees that if the constraint holds initially, it will hold throughout once we have found the actual orbit through the variational principle.

We can always add a differential term dF to the action integral and define an alternative Hamiltonian with the same dynamics. Since τ is arbitrary, we can choose a form that is convenient, for example $\tau = t$ would give $t' = 1$ and $p_t' = -\partial H/\partial t$. More generally, we can apply any canonical transformation to the $n + 1$ co-ordinates, and rewrite the constraint Eq. (1.6) in the most general way as

$$F(Q_u, P_u) = 0 \ . \tag{1.9}$$

Here, the index u ranges from 1 to $n+1$, the new phase space variables can be a complicated function of the original ones, and F will not explicitly depend on τ. The new extended Hamiltonian will have the form

$$\hat{H}_{\text{ext}}(Q_u, P_u, \tau) = \lambda(\tau) F(Q_u, P_u) \ . \tag{1.10}$$

Note that nowhere did we assume anything about the original H, and it can have arbitary time-dependence. The point is that any time-dependence can expressed in terms of t instead of τ, and mixed into other variables through canonical transformations.

The function λ is set by how we wish to define τ. One common choice is to take $\tau = Q_{n+1}$ (often you will see Q_0, where the index of u goes from 0 to n), making this co-ordinate "time-like". Being able to do this depends somewhat on the dynamical system. To have $Q'_{n+1} = 1$ requires $\lambda \equiv (\partial F/\partial P_{n+1})^{-1}$. Then we have

$$P'_{n+1} = \frac{\mathrm{d}P_{n+1}}{\mathrm{d}Q_{n+1}} = -\frac{\partial H_{\mathrm{ext}}}{\partial Q_{n+1}} = -\frac{\partial F/\partial Q_{n+1}}{\partial F/\partial P_{n+1}} \ . \tag{1.11}$$

Often it is simplest to write F so $\lambda \equiv 1$, so the extended Hamiltonian $\hat{H}_{\mathrm{ext}} = P_{n+1} + K(Q_i, P_i, Q_{n+1})$, where i ranges from 1 to n, Then $Q'_{n+1} = 1$, $P'_{n+1} = -\partial K/\partial Q_{n+1}$, and for the other degrees of freedom $Q'_i = \partial K/\partial P_i$ and $P'_i = -\partial K/\partial Q_i$.

In general, the action integral is

$$S = \int_{\tau_1}^{\tau_2} \left(\sum_{i=1}^{n+1} P_u Q'_u - \lambda F(P_u, Q_u) \right) \mathrm{d}\tau \ . \tag{1.12}$$

The constraint that $F = 0$ for the proper orbit makes the definite integral

$$\int \sum_{i=1}^{n+1} P_u \mathrm{d}Q_u \tag{1.13}$$

particularly interesting. This is one way to introduce Hamilton-Jacobi theory, by taking Q_{n+1} to be time-like and expressing the other degrees of freedom in terms of action-angle variables.

1.2 Extended generating functions and canonical transformations

There is no significant difference between canonical transformations in extended phase space versus in regular phase space. The main thing to keep in mind is that the conjugate to time is equal in value to $-H$ rather than $+H$. Often, p_t is replaced by $e = -p_t$, or possibly $h = -p_t$.

Let us focus on generating functions of the second kind, and we consider just one spatial degree of freedom for simplicity. For a given $F_2(q, t, P, E)$, we have

$$p = \frac{\partial F_2}{\partial q} \ , \qquad e = -\frac{\partial F_2}{\partial t} \ , \qquad Q = \frac{\partial F_2}{\partial P} \ , \qquad T = -\frac{\partial F_2}{\partial E} \ . \tag{1.14}$$

Returning to the regular phase space, the new Hamiltonian has to be $\hat{H} = H - e + E$ to preserve the constraint. The extended Hamiltonian can only be different by a proportionality factor because it follows the same constraint to vanish on the true orbit. Unless the independent variable τ is redefined that ratio should be 1.

1.3 Extended point transformations

Redefining the co-ordinates is more interesting in extended phase space because we can easily redefine time along with the spatial coordinates. Let us suppose we already have the usual Hamiltonian in the form $H(x, y, s, p_x, p_y, p_s, t)$ where s is a desirable choice for the independent coordinate. In extended phase space we have $-e + H$ for the extended Hamiltonian. If we want to swap s and t, we can use one of the simplest generating functions, $F_2 = xP_x + yP_y - sE + tP_s$. Then the transverse variables are unchanged and $p_s = -E$, $e = -P_s$, $S = t$ and $T = s$. Going back to regular space, the new Hamiltonian K has to satisfy $K = E = -p_s$, and so

$$H\left(x, y, s, p_x, p_y, -K\left(x, y, t, p_x, p_y, -e, s\right), t\right) - e = 0 \ . \tag{1.15}$$

We can get the motion in time by using $t' = -\partial K/\partial e$.

If H is independent of t, then so is K. This means that $e' = -\partial K/\partial t = 0$, and e is a constant of the motion. Thus we can write $K(x, y, p_x, p_y, s; e)$ for the Hamiltonian where e is just a parameter. This gives a family of solutions which appear to only have two degrees of freedom, although there is still the s-dependence in the Hamiltonian which effectively acts like an additional degree of freedom.

1.4 Lorentz transformation

First, let us consider non-relativistic mechanics and change to a reference frame moving with velocity v relative to the original frame. In this case we will keep time unchanged, $T = t$, and the new co-ordinate will satisfy $Q = q - vt$. Then we have

$$F_2(q, t, P, E) = Pq - Et - v(Pt - m_0 q) \ . \tag{1.16}$$

This yields

$$p = P + m_0 v, \qquad e = E + vP, \qquad Q = q - vt, \qquad T = t \ . \tag{1.17}$$

The extended Hamiltonian is unchanged. While the value of the regular Hamiltonian satisfies $H = \hat{H} + vP$ as required by the energy equation, its form is the same as long as any movement in for example, constraints or potentials is taken into account.

The Lorentz transformation is defined by

$$F_2(q, t, P, E) = \gamma \left[Pq - Et - v\left(Pt - \frac{Eq}{c^2} \right) \right] \ , \tag{1.18}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$ is the Lorentz factor associated with the velocity shift v. This yields

$$p = \gamma(P + Ev/c^2), \qquad e = \gamma(E + vP), \qquad Q = \gamma(q - vt), \qquad T = \gamma(t - vq/c^2) \ . \tag{1.19}$$

The conjugate momenta must always be defined in relation to the dynamical system, in particular when electromagnetic fields are present we must use $p = p_{\text{mech}} + qA/c$. It takes some work to see this, but even in this case the form of the expression for the Hamiltonian is unchanged, just the specific values are shifted and the fields have to be transformed.

Supplementary Notes 2

Fluid Equations

2.1 Deriving bulk equations from the Vlasov equation

We start from the *continuity* equation, which is in some sense the most direct way to get to the fluid equations.

$$0 = \frac{\partial f}{\partial t} + \frac{\partial}{\partial \boldsymbol{q}} \left(\dot{\boldsymbol{q}}(q,p,t)\, f \right) + \frac{\partial}{\partial \boldsymbol{p}} \left(\dot{\boldsymbol{p}}(q,p,t)\, f \right) \ . \tag{2.1}$$

For Hamiltonian flow, this yields

$$\begin{aligned} 0 &= \frac{\partial f}{\partial t} + \frac{\partial}{\partial \boldsymbol{q}} \cdot \left(\frac{\partial H}{\partial \boldsymbol{p}} f \right) - \frac{\partial}{\partial \boldsymbol{p}} \cdot \left(\frac{\partial H}{\partial \boldsymbol{q}} f \right) \\ &= \frac{\partial f}{\partial t} - \{f, H\} = \frac{df}{dt} \ , \end{aligned} \tag{2.2}$$

where the two second-derivatives of H cancel out.

To get fluid equations, we essentially take integrals, with additional multiplicative factors, of Eq. (2.1). Integrating over all of the p variables:

$$0 = \int \frac{\partial f}{\partial t} dp + \int \frac{\partial}{\partial \boldsymbol{q}} \cdot (\dot{\boldsymbol{q}}(q,p,t)\, f)\ dp + \int \frac{\partial}{\partial \boldsymbol{p}} \cdot (\dot{\boldsymbol{p}}(q,p,t)\, f)\ dp = \frac{\partial}{\partial t} \int f dp + \frac{\partial}{\partial \boldsymbol{q}} \int \dot{\boldsymbol{q}} f\, dp + 0 \ , \tag{2.3}$$

assuming that the distribution is bounded in momentum-space. We can rewrite this as the fluid continuity equation,

$$0 = \frac{\partial n}{\partial t} + \nabla_{\boldsymbol{q}} \cdot (n \boldsymbol{v}) = \frac{dn}{dt} + n \nabla_{\boldsymbol{q}} \cdot \boldsymbol{v} \ . \qquad A_q \tag{2.4}$$

This gives the standard fluid flow conservation equation, where we have defined the fluid density and flow velocity

$$n(\boldsymbol{q}, t) \equiv \int f\, dp \ , \qquad \boldsymbol{v}(\boldsymbol{q}, t) \equiv \frac{1}{n(\boldsymbol{q}, t)} \int \dot{\boldsymbol{q}} f\, dp \ , \tag{2.5}$$

and we re-use the symbol d/dt to be the *convective* derivative when acting on fluid quantities,

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \frac{\partial}{\partial \boldsymbol{q}} \ . \tag{2.6}$$

Note that although phase space flow is incompressible, the fluid density and velocity are under no such restrictions.

Additional fluid quantities will have to be defined as higher order moments are taken. For example, multiplying by one momentum component, p_i, before integrating yields

$$\begin{aligned} 0 &= \int p_i \frac{\partial f}{\partial t} dp + \int p_i \frac{\partial}{\partial \boldsymbol{q}} \cdot (\dot{\boldsymbol{q}}(q,p,t)\, f)\ dp + \int p_i \frac{\partial}{\partial \boldsymbol{p}} \cdot (\dot{\boldsymbol{p}}(q,p,t)\, f)\ dp \\ &= \frac{\partial}{\partial t} \int p_i f dp + \frac{\partial}{\partial \boldsymbol{q}} \int p_i \dot{\boldsymbol{q}} f\ dp - \int \dot{p}_i f\ dp \\ &= \frac{\partial}{\partial t} \left(n \bar{p}_i\right) + \frac{\partial}{\partial \boldsymbol{q}} \left(n \boldsymbol{v} \bar{p}_i + \int \left(p_i - \bar{p}_i(\boldsymbol{q},t)\right) (\dot{\boldsymbol{q}} - \boldsymbol{v}) f\ dp \right) - n F_i(\boldsymbol{q},t) \ , \end{aligned} \tag{2.7}$$

where we define

$$\bar{p}_i(\boldsymbol{q},t) \equiv \int p_i f\ dp \ , \qquad\qquad F_i(\boldsymbol{q},t) \equiv \frac{1}{n(\boldsymbol{q},t)} \int \dot{\boldsymbol{p}}_i f\ dp \ . \tag{2.8}$$

Some terms which average to zero have been removed to obtain this result. We can also use the fluid conservation equation to simplify things further,

$$0 = n \frac{d}{dt} \bar{p}_i + \sum_{k=1}^{N} \frac{\partial}{\partial q_k} \mathcal{P}_{k,i}(\boldsymbol{q},t) - n(\boldsymbol{q},t) F_i(\boldsymbol{q},t) \ , \tag{2.9}$$

where N is the number of degrees of freedom and the pressure tensor is defined as

$$\mathcal{P}_{k,i}(\boldsymbol{q},t) \equiv \int \left(\dot{q}_k - v_k\right) \left(p_i - \bar{p}_i\right) f\ dp \ . \tag{2.10}$$

Often, the pressure tensor is split into the isotropic scalar part,

$$p(\boldsymbol{q},t) \equiv \frac{1}{N} \sum_{i=1}^{N} \mathcal{P}_{i,i}(\boldsymbol{q},t) \ , \tag{2.11}$$

and the anisotropic part $\Pi_{i,j} = \mathcal{P}_{i,j} - p\delta_{i,j}$. Even when there is more than the scalar pressure, typically the anisotropic part is still diagonal for a weakly-interacting system. Off-diagonal elements are associated with viscosity.

This process can be continued to include the dynamics of energy and other higher-order terms. However, there will always be additional moments which will enter into the resulting expressions.

2.2 Issues with fluid equations

Obtaining a closed set of fluid equations is typically not possible except in some very special cases. However, for weakly interacting systems with a simple (e.g., Gaussian) distribution function, higher-order moments can be approximated in terms of lower moments.

Scattering is another important affect, in the sense that when short-range interactions are ignored in the Hamiltonian, this winds up having to be represented as abrupt, often random kicks which appear to shuffle around density in phase space. Ultimately, this comes from approximating phase space as having $2N$ degrees of freedom, rather than $2N$ times the number of particles. Scattering can also be generated by other elements of the system, for example electrons can be scattered by photons or vice-versa. The Vlasov equation may also have to be corrected by source or sink terms, such as an external supply of particles, reactions between particles, or some sort of decay mechanism.

In practice, the fluid equations miss may effects that are hidden in the averages. This is especially true when the system or even some subset of particles is weakly interacting. Often fixes for this involve adding extra "kinetic" terms in the conservation equations, or even using nonlocal operators (e.g., integrals).

Supplementary Notes 3

Relativity and Lorentz Transformations

3.1 Introduction

Note that this will be more of a "cheat sheet" on relativity than a full description.

For general relativity, the key focus is the geometry of space-time as defined by the infinitesimal distance element $\mathrm{d}\tau$ between two points in some arbitrarily small local neighborhood. By convention, and since we will be discussing paths of particles moving at less than the speed of light, we talk about the proper time instead of using length. For special relativity, it is still useful to think in terms of this quantity but it has a very simple form,

$$c^2\mathrm{d}\tau^2 = -(x_0 - x_1)^2 - (y_0 - y_1)^2 - (z_0 - z_1)^2 + c^2(t_0 - t_1)^2 \tag{3.1}$$

which defines Minkowski space. The geometry is flat, so there is no need to resort to infinitesimal elements.

We want this metric to be invariant under simple changes of reference frame. In particular, anything moving at the speed of light has $\mathrm{d}\tau = 0$, so the speed of light has to be invariant. Thus this metric is a shortcut to imposing such a condition.

Any purely spatial rotation or translation will preserve proper time, as will a simple time shift. For anything more complicated, only the *Lorentz transformation* will preserve proper time. The Lorentz transformation can be expressed as a group of matrices, but for simplicity let us just write the equations for a shift to a velocity v in the $\hat{z}$ direction:

$$\begin{aligned} x' &= x \\ y' &= y \\ z' &= \gamma(z - vt) \\ t' &= \gamma(t - vz/c^2) \ , \end{aligned} \tag{3.2}$$

where

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}} \ . \tag{3.3}$$

You can check for yourself that $\mathrm{d}\tau^2$ is preserved for any $v < c$. This is one example (really, the fundamental example) of a *Lorentz invariant*.

3.2 Lorentz transformations of momentum and energy

The conjugate quantities, momentum and energy, satisfy a similar invariant which is the rest mass or energy:

$$m_0^2 c^4 = e^2 - (p_x^2 + p_y^2 + p_z^2)c^2 \ . \tag{3.4}$$

Thus they also transform in a similar way,

$$\begin{aligned} p'_x &= p_x \\ p'_y &= p_y \\ p'_z &= \gamma(p_z - ve/c^2) \\ e' &= \gamma(e - vp_z) \ , \end{aligned} \tag{3.5}$$

I am using e here for energy (including the rest mass), below we will use h and $\mathcal{H}$ for the value of the original and transformed Hamiltonian. Hopefully this will avoid confusion with electric field.

Any kind of plane wave that goes like $f(\omega t - k_x x - k_y y - k_z z)$ can be written in the boosted frame as being proportional to $f(\omega' t' - k'_x x' - k'_y y' - k'_z z')$ with

$$\begin{aligned} k'_x &= k_x \\ k'_y &= k_y \\ k'_z &= \gamma(k_z - v\omega/c^2) \\ \omega' &= \gamma(\omega - vk_z) \ . \end{aligned} \tag{3.6}$$

For electromagnetic fields, this can be interpreted in terms of photons that have energy $\hbar\omega$ and momentum $\hbar\boldsymbol{k}$.

There is a particular useful formula for the Doppler shift when there is an angle θ between the wave vector of the field and the direction of the boost. Taking $k_z = (\omega/c)\cos\theta$, the inverse Lorentz transformation gives

$$\omega = \frac{\omega'}{\gamma(1 - \beta\cos\theta)} \ , \tag{3.7}$$

with $\beta \equiv v/c$. The transformed angle is given by

$$\tan\theta = \frac{\sin\theta'}{\gamma(\cos\theta' + \beta)} \ . \tag{3.8}$$

Note that for $\gamma \gg 1$, almost all angles θ' (except for those very close to π) have counterparts $\theta \simeq 1/\gamma$. When $\gamma \gg 1$ and $\theta \ll 1$, we can approximate

$$\omega \simeq \frac{2\gamma\omega'}{1 + \gamma^2\theta^2} \ . \tag{3.9}$$

For any system, we can use Eq. (3.5) to see how the total energy and momentum transform. For example, the reference frame velocity shift $\boldsymbol{v}$ which takes one into the center-of-momentum frame (where the total momentum vector vanishes) satisfies $\boldsymbol{v} = \boldsymbol{p}c^2/e$, where $\boldsymbol{p}$ is the total momentum and e is the total energy in the system.

3.3 Lorentz transformations of other physical quantities

Having defined the transformation of phase space, the transformation of almost any other quantity is easy to derive. For example, anything which produces a force (such as an electric field) gives a $\mathrm{d}\boldsymbol{p}/\mathrm{d}t$ term, and we know how momentum and time transform. Here we will just give results for various quantities, always involving a boost in the z-direction.

The electromagnetic fields transform very differently from position and momentum:

$$\begin{aligned} E_z &= E'_z \ , & \boldsymbol{E}_\perp &= \gamma\left(\boldsymbol{E}'_\perp - \boldsymbol{v}\times\boldsymbol{B}'\right) \ , \\ B_z &= B'_z \ , & \boldsymbol{B}_\perp &= \gamma\left(\boldsymbol{B}'_\perp + \frac{1}{c^2}\boldsymbol{v}\times\boldsymbol{E}'\right) \ . \end{aligned} \tag{3.10}$$

Here, $\boldsymbol{E}_\perp$ and $\boldsymbol{B}_\perp$ refer to the components of the electric and magnetic fields perpendicular to the boost velocity $\boldsymbol{v}$, so they can be taken to include the x and y components. For electromagnetic fields the Lorentz transformation has been inverted, because usually we start with the fields in the rest frame and we want to know the fields in the lab frame.

There are many other ways to describe the fields. The tensor notation is often useful, especially for expressing quantities in an invariant way. We will concentrate on the electromagnetic potentials, which form a four-vector $(\phi/c, \boldsymbol{A})$ that transforms exactly like $(ct, \boldsymbol{r})$:

$$\begin{aligned} A_x &= A'_x \\ A_y &= A'_y \\ A_z &= \gamma(A'_z + v\Phi'/c^2) \\ \phi &= \gamma(\phi' + vA'_z) \ , \end{aligned} \tag{3.11}$$

The wave equation involves an invariant functional form, the D'Alembertian, which is usually denoted as a square symbol:

$$0 = \Box f \equiv \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) f \ . \tag{3.12}$$

For electromagnetic waves we have either $\Box\boldsymbol{E} = 0$ or $\Box\boldsymbol{A} = 0$ with $\nabla\cdot\boldsymbol{A} = 0$ as well.

There are two main field invariants: $\boldsymbol{E}\cdot\boldsymbol{B}$, and the Lagrangian field density

$$\mathcal{L} = \frac{\epsilon_0}{2}\left(\boldsymbol{E}^2 - c^2\boldsymbol{B}^2\right) \ . \tag{3.13}$$

The total number of photons in a pulse is also a Lorentz invariant.

The field energy density is $S_0 = (\epsilon_0/2)\boldsymbol{E}^2 + (1/2\mu_0)\boldsymbol{B}^2$. Separating into components parallel and perpendicular to the velocity shift, as in Eq. (3.10), we find that

$$S_0 = \frac{\epsilon_0}{2}\left[\boldsymbol{E}'^2 + c^2\boldsymbol{B}'^2 + 2\frac{v^2}{c^2}\gamma^2\left(\boldsymbol{E}'_\perp{}^2 + c^2\boldsymbol{B}'_\perp{}^2\right) + 4\gamma^2\boldsymbol{v}\cdot(\boldsymbol{E}'\times\boldsymbol{B}')\right] . \tag{3.14}$$

The Poynting vector $\boldsymbol{S} = (\epsilon_0/c)\boldsymbol{E}\times\boldsymbol{B}$. The longitudinal component transforms as

$$S_z = \frac{\epsilon_0}{c}\left[\hat{z}\cdot(\boldsymbol{E}'\times\boldsymbol{B}') + \frac{v}{c^2}\gamma^2\left(\boldsymbol{E}'_\perp{}^2 + c^2\boldsymbol{B}'_\perp{}^2\right)\right] , \tag{3.15}$$

and the transverse component transforms as

$$\boldsymbol{S}_\perp = \frac{\epsilon_0}{c}\gamma\left[(\boldsymbol{E}'\times\boldsymbol{B}')_\perp - \frac{v}{c^2}E'_z\boldsymbol{E}'_\perp - vB'_z\boldsymbol{B}'_\perp\right] . \tag{3.16}$$

The combination of energy density and Poynting vector do not transform in a simple way, although the transformation expressions do include terms like S'_0, S'_z, and $\boldsymbol{S}'_\perp$. This is because they denote densities and not the total momentum or energy. They actually form one column (the time-like column) of a 4×4 tensor, the stress tensor, which is in turn formed from combinations of the electromagnetic field tensor.

When solving Maxwell's equations we may also need to transform the current and charge density:

$$\begin{aligned} j_x &= j'_x \\ j_y &= j'_y \\ j_z &= \gamma(j'_z + v\rho') \\ \rho &= \gamma(\rho' + vj'_z/c^2) . \end{aligned} \tag{3.17}$$

Total charge is a conserved quantity under Lorentz transformations.

Temperature can be a little complicated. The simplest way to consider temperature is to define it in terms of the spread in momentum in the rest frame of a fluid. This yields a comoving temperature, T_0, which by definition is an invariant. Because momentum transverse to the boost is unchanged, momentum spread transverse to the fluid motion is unchanged and the transverse temperature is $\sigma^2_{p\perp}/2m_0$. We can obtain the longitudinal temperature by looking at the derivatives of the identity $e^2 = m_0^2c^4 + c^2p_z^2$ and the transformation rule $p'_z = \gamma(p_z - ve/c^2)$. The result is $\mathrm{d}e/\mathrm{d}p_z = c^2p_z/e$ and

$$\frac{\mathrm{d}p'_z}{\mathrm{d}p_z} = \gamma\left(1 - \frac{v}{c^2}\frac{\mathrm{d}e}{\mathrm{d}p_z}\right) = \gamma\left(1 - \frac{vp_z}{e}\right) \simeq \gamma\left(1 - \frac{v\langle p_z\rangle}{\langle e\rangle}\right) \simeq \gamma\left(1 - \frac{v^2}{c^2}\right) = \frac{1}{\gamma} , \tag{3.18}$$

taking the velocity shift $v = c^2\langle p_z\rangle/\langle e\rangle$ to get into the rest frame. The longitudinal temperature is then $\sigma^2_{pz}/2m_0\gamma^2$, where γ can be taken as corresponding to the velocity of the beam in the lab frame. This is valid so long as the spread in momentum (in all directions) is much smaller than $\langle e\rangle/c \simeq (m_0^2c^2 + \langle p_z\rangle^2)^{1/2}$. For relativistic motion in the rest frame, $\langle e'\rangle - m_0c^2$ is a better measure of temperature (actually the sum of the 3 temperatures).

There is a more general formulation, that says that the inverse temperature, $\beta_{Tt} = 1/k_B T$, actually has directional components too, $\boldsymbol{\beta}_T = \boldsymbol{u}/ck_B T$, where $\boldsymbol{u}$ is the velocity of the object, and they transform by

$$\begin{aligned} \beta_{Tx} &= \beta'_{Tx} \\ \beta_{Ty} &= \beta'_{Ty} \\ \beta_{Tz} &= \gamma(\beta'_{Tz} + v\beta'_{Tt}) \\ \beta_{Tt} &= \gamma(\beta'_{Tt} + v\beta'_{Tz}/c^2) \ . \end{aligned} \tag{3.19}$$

The invariant is the inverse temperature in the rest frame, $\beta_{Tt}^2 - \boldsymbol{\beta}_T^2 = \beta_{T0}^2$. It is more natural to just talk about the invariant, but the topic of observing temperature in a moving frame has come up when analyzing the cosmic microwave background.

3.4 Relativity and classical mechanics

In the lecture notes, we wrote the Hamiltonian for the lab frame and ignored how the dynamics transforms in different reference frames. We know that electromagnetic forces do not need to be corrected even for relativistic particles, so long as we define quantities like momentum properly.

Here are a few helpful expressions and derivations for studying the Hamiltonian that is correct for special relativity. Many times you will have to take partial derivates of the gamma factor with respect to either velocity of momentum. It takes a lot of extra steps and calculations to do this by expanding out γ every time. The best thing to do is convince yourself one time that

$$\nabla_{\boldsymbol{v}} \gamma = \frac{\boldsymbol{v}}{c^2} \gamma^3 \tag{3.20}$$

or, equivalently,

$$\nabla_{\boldsymbol{v}} \left(\frac{1}{\gamma} \right) = -\frac{\boldsymbol{v}}{c^2} \gamma \ . \tag{3.21}$$

If you are taking the derivative with respect to something other than velocity, call it Q, then you simply have to take the vector product of this quantity with $\partial \boldsymbol{v}/\partial Q$.

We need the Lorentz transformation for both coordinates (t, x, y, z) and conjugate momentum (h, p_x, p_y, p_z) to define the extended canonical transformation that corresponds to the Lorentz transformation in extended phase space. This extended transformation has generating function

$$F_2(x, y, z, t, \Pi_x, \Pi_y, \Pi_z, \mathcal{H}) = \gamma \left[x\Pi_x + y\Pi_y + z\Pi_z - t\mathcal{H} - \boldsymbol{v} \cdot \left(t\boldsymbol{\Pi} - \boldsymbol{r}\mathcal{H}/c^2 \right) \right] \ . \tag{3.22}$$

The velocity v only refers to the relative velocity of the new reference frame with respect to the old one, and $\gamma = (1 - v^2/c^2)^{-1/2}$ as before.

The Hamiltonian keeps its form under Lorentz transformations. It is best to verify this by looking at the electromagnetic potential fields. Even more simply, we can ignore fields altogether, and the modified Hamiltonian satisfies $H = \gamma \left(H' + \boldsymbol{v} \cdot \Pi \right)$. This matches the condition on energy, $h = \gamma(\mathcal{H} + \boldsymbol{v} \cdot \Pi)$.

Supplementary Notes 4

Gaussian Laser Beams

4.1 Wave equation

The wave equation for a charge- and current-free region can be expressed as

$$\Box \boldsymbol{E} \equiv \nabla^2 \boldsymbol{E} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0 \ , \tag{4.1}$$

with the constraint that $\nabla \cdot \boldsymbol{E} = 0$. This is the d'Alembert equation and the functional form $\Box$ is often called the d'Alembertian operator. The magnetic field is given by

$$-\frac{1}{c}\frac{\partial B}{\partial t} = \nabla \times \boldsymbol{E} \ , \tag{4.2}$$

with the constraint that $\nabla \cdot \boldsymbol{B} = 0$.

Often this is expressed in terms of the vector potential $\boldsymbol{A}$, because it gives a more natural way to impose the constraints in the fields. The wave equation itself has the same form,

$$\Box \boldsymbol{A} = \nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = 0 \ . \tag{4.3}$$

We have some freedom in choosing the potentials, and we can require the scalar potential $\Phi = 0$ and as well choose $\nabla \cdot \boldsymbol{A} = 0$. This gauge is specific to a particular co-ordinate system, we could alternatively use the Lorentz gauge with the invariant condition

$$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \nabla \cdot \boldsymbol{A} = 0 \ . \tag{4.4}$$

This yields a slightly more complicated equation (though it is still simple in the language of the electromagnetic field tensor). We will not try to express things in an invariant way, as we are going to take more simplifying assumptions anyhow.

4.2 Eikonal approximation

We are going to assume that the fields we are interested in are in some sense close to a specific plane wave. The nominal direction of propagation will be taken to be in the z-direction. Thus, we are going to approximate $\boldsymbol{A} \simeq \boldsymbol{a} \exp(-i\omega_0 t + ik_0 z)/2 + \text{c.c.}$, where c.c. refers to the complex conjugate. The eikonal vector $\boldsymbol{a}$ will in general be complex valued. The frequency ω_0 and wave vector k_0 are required to be real valued, and we need to have $\omega_0 = ck_0$ in a vacuum.

What exactly do we mean by approximate? We will allow the parameters which define this plane wave to change, but only slowly. In its most general form, we can take

$$\boldsymbol{A} \simeq \frac{1}{2}\boldsymbol{a}(x,y,z,t) \exp\left[-i\omega(x,y,z,t)t + i\boldsymbol{k}(x,y,z,t)\cdot\boldsymbol{r}\right] + \text{c.c.} \ . \tag{4.5}$$

There will be some constraints relating frequency and wave vector, which we will actually alter below to allow for *weak* corrections due to dielectrics or plasmas, and also constraints on the amplitude and direction of $\boldsymbol{a}$. We could even allow for damping or growth of the waves in a medium, but for slow damping it is more natural to fold this into the factor $\boldsymbol{a}$.

If we assume that all spatial derivatives of $\boldsymbol{a}$, ω, and $\boldsymbol{k}$ are small compared to the range of wave vectors we care about, for example,

$$|\nabla\omega| \ll |\boldsymbol{k}\omega| \ , \tag{4.6}$$

and similarly that the time derivatives are small compared to the range of frequencies we care about,

$$\left|\frac{\partial\omega}{\partial t}\right| \ll \left|\omega^2\right| \ . \tag{4.7}$$

This means that not only does the pulse have to be much wider than the characteristic wavelength, but it has to have a duration that is many periods long. In fact, short laser pulses are extremely difficult to express analytically, even approximately, except for the case of a pure plane wave. At this point, we have something roughly similar to a ray-tracing algorithm.

4.3 Paraxial approximation

Now as long as $\omega = c|\boldsymbol{k}|$, the above expressions can be made to give some simplifying results at least over some region in space and time. We are going to go one step further and take the *paraxial* approximation that the frequency and direction of the radiation never vary enough to have to worry about one frequency component changing quickly relative to another frequency component. In practice, this just means that the relative bandwidth of the pulse much be much smaller than unity, and the radiation pulse cannot be changing direction significantly (some of the corrections we we consider below could allow a large bending angle over a large distance). It also means that the angular divergence of the pulse must be small.

So we will just set $\omega \equiv \omega_0$, $\boldsymbol{k} \equiv k_0 \hat{z}$, and of course $\omega_0 = ck_0$. Any additional phase variation will be folded into the amplitude $\boldsymbol{a}$. Something like a pre-defined frequency chirp that is itself slow but continues long enough to lead to a large could be handled explicitly, that will not be considered here.

The eikonal field is then

$$\boldsymbol{A} \simeq \frac{1}{2}\boldsymbol{a}(x,y,z,t)\exp\left(-i\omega_0 t + ik_0 z\right) + \text{c.c.} \ . \tag{4.8}$$

All slow variations are folded into $\boldsymbol{a}$, which may for example included changes such as phase oscillations in time if the frequency is slightly shifted.

The eikonal equation expands out to

$$0 = \Box\boldsymbol{A} \simeq \left[\left(\frac{\omega_0^2}{c^2} - k_0^2\right)\boldsymbol{a} + 2i\left(\frac{\omega_0}{c^2}\frac{\partial \boldsymbol{a}}{\partial t} + k_0\frac{\partial \boldsymbol{a}}{\partial z}\right) - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{a}}{\partial t^2} + \nabla^2\boldsymbol{a}\right]\ \exp(-i\omega_0 t + ik_0 z) + \text{c.c.} \ . \tag{4.9}$$

To be valid everywhere, the term in brackets must vanish. We can now look at terms order by order in terms of comparing the frequency to time derivatives, and wave vector to spatial derivatives. The leading order term is simply $\omega_0^2/c^2 - k_0^2$. We eliminate this term by making sure that the magnitude of the wave vector equals ω_0/c by definition.

We could keep all of the remaining terms, and this will actual capture some interesting physical effects. However, to simplify things further, one normally throws out quantities which are much smaller than other terms above. However, we have to be careful about how this is done. The second derivative in time should be much smaller than $i\omega_0\partial\boldsymbol{a}/\partial t$. Because the term we are ignoring is in phase with the nominal field, there are not even any subtle effects which might be obscured by this approximation. However, while we know that $\partial^2\boldsymbol{a}/\partial z^2$ is small compared to $ik_0\partial\boldsymbol{a}/\partial z$, there is no direct comparison to $\nabla_\perp^2\boldsymbol{a} = \partial^2\boldsymbol{a}/\partial x^2 + \partial^2\boldsymbol{a}/\partial y^2$. So this term should be kept. One aspect of the paraxial approximation is that transverse scale lengths are substantially shorter than longitudinal scale lengths.

Dropping the smaller terms and dividing the whole expression by $2ik_0$ yields

$$\frac{\partial \boldsymbol{a}}{\partial z} + \frac{\omega_0}{k_0 c^2}\frac{\partial \boldsymbol{a}}{\partial t} - \frac{i}{2k_0}\nabla_\perp^2\boldsymbol{a} \simeq 0 \ . \tag{4.10}$$

This expression leaves out some physical effects such as group velocity changes related to the fact that total path length of the pulse is a function of z and radius. But usually this is an excellent first approximation. Note that $\omega_0/k_0 c$ has already been defined to be unity.

We also have to include the constraint that $\nabla \cdot \boldsymbol{A} = 0$. This simplifies to

$$ik_0 a_z + \nabla \cdot \boldsymbol{a} = 0 \ . \tag{4.11}$$

Because this expression is dominated by k_0, a_z has to almost vanish, and we can ignore a_z in the second term to yield

$$a_z \simeq \frac{i}{k_0}\nabla_\perp \cdot \boldsymbol{a}_\perp \ . \tag{4.12}$$

Thus Eq. (4.10) should only be applied to the transverse components of the vector potential. The electric field is given by

$$\begin{aligned} \boldsymbol{E} &= -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} \\ &= -\frac{1}{2c}\left[\left(-i\omega_0 \boldsymbol{a} + \frac{\partial \boldsymbol{a}}{\partial t}\right)\exp(-i\omega_0 t + ik_0 z) + \text{c.c.}\right] \ . \end{aligned} \tag{4.13}$$

Roughly, we could just take the first term and the electric field is proportional to the vector potential. However, there are important implications to the full expression, in particular the electric field observed at a fixed point has to integrate out to zero as a pulse passes by. This is an important constraint that governs how laser pulses interact with particles, and is particularly important for schemes involving particle acceleration.

If we now assume a Gaussian profile for the laser beam (higher order modes could be considered as well), we can get a good guess for what the radiation pulse looks like, especially the transverse components. The transverse eikonal field is still a vector but the two polarizations decouple, so we just write it as a scalar. The term $\frac{i}{2k_0}\nabla_\perp^2 a$ should be weak because it has k_0 in the denominator. Thus, we know we are looking at a correction to the 1D equation $a = f(z-ct)$. A good way to parametrize the complete solution including transverse effects looks very similar to the formalism used for particle beams,

$$a(x,y,z,t) = a_0 f(z-ct) R(z) \exp\left[i\delta(z)\right] \exp\left[-k_0 x^2 \frac{1+i\alpha_x(z)}{2\beta_x(z)} - k_0 y^2 \frac{1+i\alpha_y(z)}{2\beta_y(z)}\right] \ . \tag{4.14}$$

For simplicity, let's assume cylindrical symmetry,

$$a(x,y,z,t) = a_0 f(z-ct) R(z) \exp\left[i\delta(z)\right] \exp\left[-k_0 (x^2+y^2) \frac{1+i\alpha(z)}{2\beta(z)}\right] \ . \tag{4.15}$$

The quantities R, δ, α, and β are all required to be real-valued. Otherwise we could reduce this to two functions, but separating terms would be a little more complicated. The time derivative cancels out by virtue of how f is written, so what remains is

$$\frac{1}{a}\frac{\partial a}{\partial z} + \frac{1}{ac}\frac{\partial a}{\partial t} = \frac{R'}{R} + i\delta' - \frac{k_0 r^2}{2\beta^2}\left[-\beta'(1+i\alpha) + i\alpha'\beta\right] \ . \tag{4.16}$$

We have written $R' = dR/dz$. This has to be compared with the transverse gradient term

$$\frac{i}{2k_0}\frac{1}{a}\nabla_\perp^2 a = ik_0 r^2 \frac{(1+i\alpha)^2}{2\beta^2} - i\frac{1+i\alpha}{\beta} \ . \tag{4.17}$$

These equations can be grouped into four independent equations by taking real and imaginary parts, and terms proportional to r^2 versus independent of radius.

Real, independent of radius:

$$\frac{R'}{R} = \frac{\alpha}{\beta} \ . \tag{4.18}$$

Imaginary, independent of radius:

$$\delta' = -\frac{1}{\beta} \ . \tag{4.19}$$

Real, quadratic in radius:

$$\alpha = -\beta'/2 \ . \tag{4.20}$$

Imaginary, quadratic in radius: using $\alpha = -\beta'/2$, this simplifies to

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}(\beta')^2 - 1 = 0 \ . \tag{4.21}$$

In addition, $R'/R = -\beta'/2\beta$, which implies that $R(z) \propto \beta^{-1/2}$. This is consistent with the conservation of energy flux (which goes like $|a|^2$) across different planes in z. The equation for the beta function is similar to that for a particle beam without any focusing. The solution has a minimum beta function β_0, corresponding to a minimum rms spot size in *intensity* (which goes like $|a^2|$) $\sigma_{x0}^2 = \sigma_{y0}^2 = \beta_0/2k_0$. This definition of β_0 winds up being identical to the Rayleigh range Z_R. The radiation pulse cross-section (again, in terms of intensity) matches that of a particle beam with geometric emittance equal to $1/2k_0$. In addition, if one follows a particular region within the pulse as it propagates, there is a phase shift of π going from one side of the region of minimum cross section to the other. You can think of half of this phase shift coming from each of the horizontal and vertical contributions.

This is not the most general functional form possible, besides changes in frequency over time one could have movement in the radiation waist location or other variations, but it is the most appropriate for a basic well-controlled laser pulse.

4.4 Transverse focusing

For propagation in a vacuum with no boundary conditions, there will not be any way to alter the process of diffraction. Waveguides are one means to guide a radiation pulse transversely. Alternatively, a cylindrical channel with any radial variation that affects the electromagnetic fields can be used to generate some focusing. The two main methods to accomplish this, a dielectric with a gradient in dielectric parameter or a plasma with a density gradient, work in essentially the same way.

A dielectric with index of refraction n typically has dielectric parameter $\epsilon = n^2\epsilon_0$. We will assume this has a transverse gradient. Similarly, a plasma has dielectric parameter $\epsilon_0(1 - \omega_p^2/\omega^2)$. We can assume the frequency is roughly ω_0. The plasma frequency ω_p goes like the square root of the density, so a density gradient will yield a focusing effect as well. If the dielectric parameter is very close to unity, we can just take this as another perturbative term. Otherwise, as long as the variation is slow and over a narrow interval, the only correction is that we can no longer assume $\omega_0 = ck_0$. Instead, we should use the corresponding dispersion equations, either $\omega_0 = ck_0/n_0$ for a dielectric or $\omega_0^2 = c^2k_0^2 + \omega_{p0}^2$. In both of these cases it is natural to take the nominal value on axis as a reference point,

and take $\epsilon = \epsilon_1[1 + M(z) + r^2/L^2(z)]$. The quantity ϵ_1 is taken as a constant, and we should have $|M| \ll 1$ to keep things simple. This yields two extra terms in the wave equation,

$$\nabla^2 \boldsymbol{A} - \frac{\epsilon_1}{\epsilon_0 c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = \frac{1}{c^2}\left[M(z) + \frac{r^2}{L^2(z)}\right]\frac{\partial^2 \boldsymbol{A}}{\partial t^2} \ . \tag{4.22}$$

The first correction modifies the dispersion equation, and the second term gives a possibly varying focusing term. In terms of the expression for a (choosing one perpendicular component) we now have

$$\frac{\partial a}{\partial z} + \frac{\omega_0 \epsilon_1}{\epsilon_0 k_0 c^2}\frac{\partial a}{\partial t} - \frac{i}{2k_0}\nabla_\perp^2 a \simeq i\frac{\omega_0^2 \epsilon_1}{2\epsilon_0 k_0 c^2}\left[M(z) + \frac{r^2}{L^2(z)}\right] a \ . \tag{4.23}$$

There should normally be no need to simultaneously consider deviations in the dielectric parameter and derivatives of the eikonal field. Higher order derivatives are required to get the group velocity correct.

The only changes to the evolution of a Gaussian pulse are in the equations for δ and β. We now have

$$\delta' = -\frac{1}{\beta} - \frac{1}{2}k_0 M(z) \ , \tag{4.24}$$

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}(\beta')^2 + \beta^2\frac{1}{L^2(z)} - 1 = 0 \ . \tag{4.25}$$

This is exactly the same as the equation for the betatron function for particle beams if we identify $K(z) = 1/L^2(z)$. If the radiation is moving along a path that is not quite straight, it is desirable to switch to a path length coordinate s just as for a particle beam.

For a constant parabolic profile L, we can have an equilibrium solution with $\beta \equiv L$. If the beam is mismatched, there would be an oscillation between a minimum and maximum β so as to satisfy $\beta_{\min}\beta_{\max} = L^2$. In terms of the rms power spot size $\sigma_x = \sigma_y = \sqrt{\beta/2k}$, this yields

$$\sigma_{x,\min}\sigma_{x,\max} = \frac{L}{2k_0} \ . \tag{4.26}$$

4.5 More conventional parametrization

The parametrization used here can be replaced by the Rayleigh range Z_R, which is the beta function at the waist, and the value of z where the waist is located, denoted s_w. The resulting formula for the eikonal field in the case of cylindrical symmetry is

$$a(x, y, z, t) = a_0 f(z - ct)\frac{Z_R}{Z_R + i(z - s_w)}\exp\left[-\frac{1}{2}\frac{k_0(x^2 + y^2)}{Z_R + i(z - s_w)}\right] \tag{4.27}$$

These parameters Z_R and s_w have the advantage of being constant while the there is no applied focusing of the radiation field. On the other hand, they are not as useful when tracking a radiation field through multiple lenses. The earlier parametrization matches that used for particle beams, and has the same nice properties of defining the transport through an optical beamline without needing to completely redo the calculation for different initial conditions.

Supplementary Notes 5

Synchrotron Radiation Damping and Heating

5.1 Classical synchrotron radiation damping

The total radiation produced by an electron moving through a transverse magnetic field is [1]

$$P_s = \frac{2}{3} r_e m_e c^2 \left(\frac{v}{c}\right)^4 \gamma^4 \frac{1}{\rho^2} \,. \tag{5.1}$$

Here, r_e is the classical electron radius, m_e is the electron mass, v is the electron velocity, c is the speed of light, $\gamma m_e c^2$ is the total electron energy, $1/\rho = ec^2 B_\perp/(m_e v\gamma)$ is the inverse radius of curvature, and $B_\perp$ is the transverse magnetic field.

The main effect is for the electron to lose energy, and the average energy loss will then be compensated for in some way. There is a cooling effect on this given by

$$\frac{1}{\sigma_E}\frac{\mathrm{d}\sigma_E}{\mathrm{d}t} = -\frac{\mathrm{d}P_s}{\mathrm{d}E} \,. \tag{5.2}$$

This can be seen by considering two particles with energy E and $E+\delta$, for small δ and time step the first one ends with energy $E - P_s(E)t$ while the second ends with $E+\delta-P_s(E+\delta)t \simeq E - P_s(E)t - \delta t[\mathrm{d}P_s(E)/\mathrm{d}E]$. The energy difference after a time t is smaller by a factor of $1 - t[\mathrm{d}P_s(E)/\mathrm{d}E]$.

In terms of longitudinal emittance, this gives

$$\frac{1}{\epsilon_z}\frac{\mathrm{d}\epsilon_z}{\mathrm{d}t} = -\frac{\mathrm{d}P_s}{\mathrm{d}E} \,, \tag{5.3}$$

which is more appropriate for taking averages over many turns because there will usually be some dynamics mixing energy and time or position within the bunch. This *synchrotron motion* is usually driven by RF acceleration which generates a potential well for the electrons if they are timed so as to miss the peak electric field.

Because the radius of curvature scales linearly with energy, for large γ and $v \simeq c$ the total radiated power scales as the square of the energy. Thus $\mathrm{d}P_s/\mathrm{d}E \simeq 2P_s/E$.

This is not the only effect of synchrotron radiation, however. Because the radiation reaction force is oriented opposite to the velocity of the particle, there are transverse components to the damping force as well. Taking $\boldsymbol{F}_r$ parallel to the momentum vector $\boldsymbol{p}$, we have

$$\boldsymbol{p} \cdot \boldsymbol{F}_r = \boldsymbol{p} \cdot \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}p^2}{\mathrm{d}t} = \frac{1}{2c^2}\frac{\mathrm{d}E^2}{\mathrm{d}t} = \frac{E}{c^2}\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{E}{c^2}P_s \ , \tag{5.4}$$

and so

$$\boldsymbol{F}_r = -\boldsymbol{p}\frac{E^2}{c^2p^2}\frac{P_s}{E} \ . \tag{5.5}$$

Thus each component of the transverse momentum shrinks,

$$\frac{1}{p_x}\frac{\mathrm{d}p_x}{\mathrm{d}t} \simeq -\frac{P_s}{E} \ , \qquad\qquad \frac{1}{p_y}\frac{\mathrm{d}p_y}{\mathrm{d}t} \simeq -\frac{P_s}{E} \ . \tag{5.6}$$

For large γ the ratio E/cp is very close to 1. In terms of momentum spread, $\mathrm{d}\sigma_{px}/\mathrm{d}t = -\sigma_{px}P_s/E$. Note that the transverse momentum damping rate is half that of the energy. Any effect on transverse momentum from the variation of power radiated with energy is negligible, because it takes a very large transverse momentum to have an impact on the total energy. The transverse emittances have the same damping,

$$\frac{1}{\epsilon_x}\frac{\mathrm{d}\epsilon_x}{\mathrm{d}t} = -\frac{P_s}{E} \ , \qquad\qquad \frac{1}{\epsilon_y}\frac{\mathrm{d}\epsilon_y}{\mathrm{d}t} = -\frac{P_s}{E} \ . \tag{5.7}$$

5.2 Beam phase space damping rates

Although these damping rates are fine for instantaneous changes, when taking the average it is more appropriate to only look at the emittances.

Finally, there can be coupling between the various degrees of freedom, so that the emittance changes are not exactly what was calculated. This occurs because the action coordinates associated with transverse motion may have some dependence on energy and time. Instead, it is best to look at the phase space density f or the combined 6D emittance, $\epsilon_{6\mathrm{D}} \equiv \epsilon_x\epsilon_y\epsilon_z$. According to the Liouville theorem, the phase space density is conserved by Hamiltonian processes, so only the radiation damping will change this quantity. Thus, instantaneous changes in f can be integrated across the ring in a trivial way, regardless of how complicated the Hamiltonian motion is or what the action coordinates look like.

We can directly obtain the equation for the 6D emittance,

$$\frac{1}{\epsilon_{6\mathrm{D}}}\frac{\mathrm{d}\epsilon_{6\mathrm{D}}}{\mathrm{d}t} = \frac{1}{\epsilon_x}\frac{\mathrm{d}\epsilon_x}{\mathrm{d}t} + \frac{1}{\epsilon_y}\frac{\mathrm{d}\epsilon_y}{\mathrm{d}t} + \frac{1}{\epsilon_z}\frac{\mathrm{d}\epsilon_z}{\mathrm{d}t} = -4\frac{P_s}{E} \ . \tag{5.8}$$

This expression, known as Robinson's sum rule [2], remains true regardless of what kind of coupling is in the ring, even if the individual emittances have different damping rates than predicted above.

The equation for the phase space density f is essentially the same, it just has the opposite sign:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 4\frac{P_s}{E}f \ . \tag{5.9}$$

For the average damping rate over a ring with circumference C, it is simply necessary to replace P_s with

$$\bar{P}_s \equiv \frac{1}{C}\oint P_s(s)\,\mathrm{d}s \ . \tag{5.10}$$

5.3 Spectrum and fluctuations of incoherent synchrotron radiation

The fluctuation in number of photons due to incoherent synchrotron radiation from an individual electron will follow a Poisson distribution, so that

$$\sigma_N \equiv \left(\langle N_{ph}^2\rangle - \langle N_{ph}\rangle^2\right)^{1/2} = \langle N_{ph}\rangle^{1/2} \ , \tag{5.11}$$

where N_{ph} is the number of photons emitted in one instance, and $\langle N_{ph}\rangle$ is the expectation value of the photon number. Because emission probabilities at different times and at different frequencies are essentially uncorrelated, the fluctuations add in quadrature, yielding

$$\sigma_N^2 = N_{ph} = \int \mathrm{d}t\,\frac{\mathrm{d}N_{ph}}{\mathrm{d}t} = \int \mathrm{d}t \int \mathrm{d}\omega\,\frac{\mathrm{d}^2N_{ph}}{\mathrm{d}t\,\mathrm{d}\omega} \ , \tag{5.12}$$

where $\mathrm{d}N_{ph}/\mathrm{d}t$ and $\mathrm{d}^2N_{ph}/\mathrm{d}t\,\mathrm{d}\omega$ are the expectation values of the photon emission rate and photon emission spectrum respectively.

The same statistics apply to the fluctuation in energy loss $\sigma_{\Delta E}$, with the energy per photon equal to $\hbar\omega$:

$$\sigma_{\Delta E}^2 \equiv \langle(\Delta E)^2\rangle - \langle\Delta E\rangle^2 = \int \mathrm{d}t \int \mathrm{d}\omega\,(\hbar\omega)^2\frac{\mathrm{d}^2N_{ph}}{\mathrm{d}t\,\mathrm{d}\omega} = \int \mathrm{d}t \int \mathrm{d}\omega\,(\hbar\omega)\frac{\mathrm{d}P}{\mathrm{d}\omega} \ , \tag{5.13}$$

where ΔE is the energy loss in one instance, and $\mathrm{d}P/\mathrm{d}\omega \equiv (\hbar\omega)\mathrm{d}^2N_{ph}/\mathrm{d}t\,\mathrm{d}\omega$ is the expectation value of the power spectrum. The expectation value of the energy loss itself is

$$\langle\Delta E\rangle = -\int \mathrm{d}t \int \mathrm{d}\omega\,\hbar\omega\frac{\mathrm{d}^2N_{ph}}{\mathrm{d}t\,\mathrm{d}\omega} = -\int \mathrm{d}t \int \mathrm{d}\omega\,\frac{\mathrm{d}P}{\mathrm{d}\omega} \, . \tag{5.14}$$

So long as the critical frequency $\omega_c = (3/2)c\gamma^3/\rho$ satisfies $\hbar\omega_c \ll \gamma m_e c^2$ [3, 4], we can use the classical result for radiation in a magnetic field,

$$P_s \equiv \int \mathrm{d}\omega\,\frac{\mathrm{d}P}{\mathrm{d}\omega} = \frac{2}{3}r_e m_e c^2\left(\frac{v}{c}\right)^4\gamma^4\frac{1}{\rho^2} \, . \tag{5.15}$$

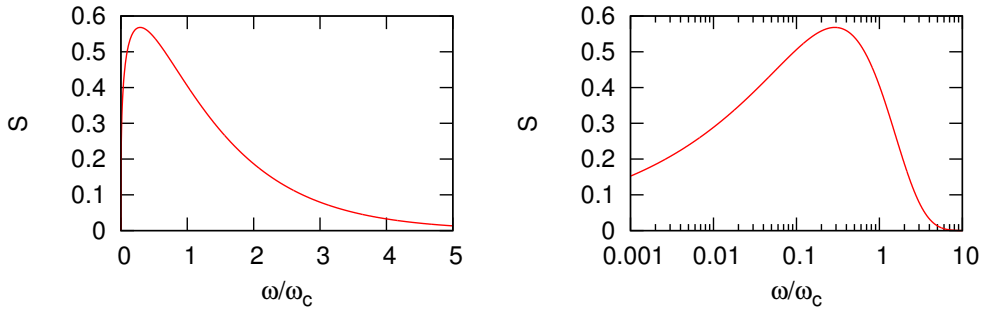

Figure 5.1: The universal spectrum function S, plotted on a regular and log scale.

We can also use the classical power spectrum

$$\frac{\mathrm{d}P}{\mathrm{d}\omega} = \frac{P_s}{\omega_c} S\left(\frac{\omega}{\omega_c}\right) \ , \tag{5.16}$$

where S is known as the universal spectral function,

$$S(\xi) \equiv \frac{9\sqrt{3}}{8\pi}\xi \int_\xi^\infty \mathrm{d}x\, K_{5/3}(x) \ , \tag{5.17}$$

and $K_{5/3}$ is a modified Bessel function. $S(\xi)$ is normalized such that $\int \mathrm{d}\xi\, S(\xi) = 1$. Upon integrating by parts, the normalization is consistent with $\int_0^\infty dx\, (x^2/2) K_{5/3}(x) = 8\pi/(9\sqrt{3})$. The spectrum decays strongly for $\omega \gg \omega_c$, so in this regime we can ignore any corrections which start to kick in for photon energies comparable to the energy of the electrons. The function is plotted in Fig. 5.1.

Taking $\gamma \gg 1$ and $v \simeq c$, the standard deviation of the energy loss is given by

$$\sigma_{\Delta E}^2 = \int \mathrm{d}t \int \mathrm{d}\omega\, (\hbar\omega) \frac{\mathrm{d}P}{\mathrm{d}\omega} = \int \mathrm{d}t\, (\hbar\omega_c) P_s \int \mathrm{d}\xi\, \xi S(\xi) = \frac{55}{24\sqrt{3}} \int \mathrm{d}t\, \hbar\omega_c P_s \ . \tag{5.18}$$

The integral $\int \mathrm{d}\xi\, \xi S(\xi)$ simplifies through integration by parts to

$$\frac{9\sqrt{3}}{8\pi} \int_0^\infty \mathrm{d}x \frac{x^3}{3} K_{5/3}(x) = \frac{9\sqrt{3}}{8\pi} \frac{55\pi}{81} = \frac{55}{24\sqrt{3}} \ . \tag{5.19}$$

To summarize, the rate of energy spread growth due to quantum heating, assuming the average energy loss is being compensated for, is given by

$$\frac{\mathrm{d}\sigma_E^2}{\mathrm{d}t} = \frac{55}{24\sqrt{3}} \hbar\omega_c P_s \ , \tag{5.20}$$

where P_s and ω_c are local quantities. The average over a ring will still have to be taken at some point to get the effective damping rate.

In terms of the phase space density f, the incoherent nature of this process can be glossed over by describing this as a diffusion process, where

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}\sigma_E^2}{\mathrm{d}t}\frac{\partial^2 f}{\partial E^2} = \frac{55}{48\sqrt{3}}\hbar\omega_c P_s \frac{\partial^2 f}{\partial E^2} \ . \tag{5.21}$$

The sign implies that the distribution function decreases where f is at a maximum.

In the absence of coupling, the net equation for σ_E is

$$\frac{\mathrm{d}\sigma_E^2}{\mathrm{d}t} = -2\sigma_E^2\frac{\mathrm{d}P_s}{\mathrm{d}E} + \frac{55}{24\sqrt{3}}\hbar\omega_c P_s = -\left(4\frac{\sigma_E^2}{E} - \frac{55}{24\sqrt{3}}\hbar\omega_c\right)P_s \ , \tag{5.22}$$

and in terms of the longitudinal emittance we have

$$\frac{1}{\epsilon_z}\frac{\mathrm{d}\epsilon_z}{\mathrm{d}t} = -\left(2 - \frac{55}{48\sqrt{3}}\frac{\hbar\omega_c E}{\sigma_E^2}\right)\frac{P_s}{E} \ . \tag{5.23}$$

Note that the average of the heating is slightly complicated because we have to average over $\omega_c P_s$ and not just P_s as before. The factor ω_c brings in an extra factor of $1/\rho$, so it is generally better to have weaker magnetic fields and softer bends. Below, we just redefine ω_c so that $\omega_c \bar{P}_s$ gives the correct average.

5.4 Equilibrium emittances

As discussed earlier, we know that there will be some synchrotron motion coupling energy and the time coordinate (or position within the bunch), so the longitudinal emittance is a more appropriate quantity to focus on than the energy spread. But there is an unspecified amount of coupling between the energy and other degrees of freedom, so we should really look at evolution of the product of the emittances,

$$\frac{1}{\epsilon_{6\mathrm{D}}}\frac{\mathrm{d}\epsilon_{6\mathrm{D}}}{\mathrm{d}t} = -\left(4 - \frac{55}{48\sqrt{3}}\frac{\hbar\omega_c E}{\sigma_E^2}\right)\frac{P_s}{E} \ . \tag{5.24}$$

This can also be described in terms of the phase space density,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \left(4f + \frac{55}{48\sqrt{3}}\hbar\omega_c E\frac{\partial^2 f}{\partial E^2}\right)\frac{P_s}{E} \ . \tag{5.25}$$

If we ignore coupling altogether, the equilibrium energy spread will be given by

$$\frac{\sigma_{E0}}{E} = \left(\frac{55}{96\sqrt{3}}\frac{\hbar\omega_c}{E}\right)^{1/2} \ . \tag{5.26}$$

The bunch length will simultaneously evolve until reaching equilibrium, but that is set by the above condition on energy spread and the synchrotron motion. In such a case, the transverse

degrees of freedom will not reach an equilibrium until other effects come into play to counter the transverse damping.

In the presence of coupling, not only will the damping terms change but new heating terms will appear in the expressions for transverse emittances. A careful definition of the horizontal action will contain terms related to the energy offset, so scattering in energy will be associated with scattering in horizontal action as well. Thus, the 6D phase space equations need to be modified to include more heating terms even when the synchrotron radiation, which is the source of the heating, remains unchanged. These will set the equilibrium horizontal and vertical emittances.

The equilibrium energy spread itself should only be slightly affected if the coupling is weak, because energy offsets are already a major aspect of the longitudinal action. However, the horizontal emittance in particular is coupled to energy through the presence of dispersion (and its derivative) in the dipoles. As electrons lose variable amount of energy, their nominal orbits jump around and this drives horizontal emittance growth.

The equilibrium horizontal size typically scales as

$$\sigma_{x0} \simeq |\eta_x| \, \frac{\sigma_{E0}}{E} \ , \tag{5.27}$$

where η_x is the horizontal dispersion. This is typically < 1 m, which can yield spot sizes of 100 micron or smaller, and normalized emittances of 1 μm. One way to get around this is to have most of the radiated power come from wigglers (with strong, oscillating transverse fields) instead of dipole bends. As long as the beam transport is designed so that the dispersion in the wigglers is weak, there will be very little heating of the horizontal emittance. However, the bends are still there (these are rings after all), and even if the dipole fields are made quite weak and the circumference of the ring is made very large, there will still be a significant equilibrium horizontal emittance.

The vertical emittance can be much smaller if there are no vertical bends. Alignment errors and accidental coupling will yield some finite residual vertical emittance. In principle, there are some transverse kicks due to quantum effects in the same way that there are energy kicks. However, the radiation is already in a narrow cone with angles on the order of $1/\gamma$, and it would be quite unusual for the vertical emittance to approach a quality where such effects matter. Furthermore, the fact that the horizontal emittance is much larger means that often the vertical emittance is completely ignorable.

Bibliography

[1] *Handbook of Accelerator Physics and Engineering*, A.W. Chao and M. Tigner, eds.; World Scientific **1999**.

[2] K.W. Robinson *PR* **1958**, *111*, 373.

[3] J. Schwinger, *Proc. N. A. S.* **1954**, *40*, 132–136.

[4] *A Quantum Legacy: Seminal Papers of Julian Schwinger*, Kimball A. Milton, ed.; World Scientific **2000**.